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Applicability of Kolmogorov's and Monin's equations of turbulence

By **REGINALD J. HILL**

National Oceanic and Atmospheric Administration, Environmental Research Laboratories, Environmental Technology Laboratory, Boulder, CO 80303-3328, USA

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The equation relating second- and third-order velocity structure functions was presented by Kolmogorov; Monin attempted to derive that equation on the basis of local isotropy. Recently, concerns have been raised to the effect that Kolmogorov's equation and an ancillary incompressibility condition governing the third-order structure function were proven only on the restrictive basis of isotropy and that the statistic involving pressure that appears in the derivation of Kolmogorov's equation might not vanish on the basis of local isotropy. These concerns are resolved. In so doing, results are obtained for the second- and third-order statistics on the basis of local homogeneity without use of local isotropy. These results are applicable to future studies of the approach toward local isotropy. Accuracy of Kolmogorov's equation is shown to be more sensitive to anisotropy of the third-order structure function than to anisotropy of the second-order structure function. Kolmogorov's 4/5 law for the inertial range of the third-order structure function is obtained without use of the incompressibility conditions on the second- and third-order structure functions. A generalization of Kolmogorov's 4/5 law, which applies to the inertial range of locally homogeneous turbulence at very large Reynolds numbers, is shown to also apply to the energy-containing range for the more restrictive case of stationary, homogeneous turbulence. The variety of derivations of Kolmogorov's and Monin's equations leads to a wide range of applicability to experimental conditions, including, in some cases, turbulence of moderate Reynolds number.

1. Introduction

Kolmogorov (1941) obtained the dynamic equation

$$D_{111}(r) = 6\nu \frac{\mathrm{d}D_{11}(r)}{\mathrm{d}r} - \tfrac{4}{5}\epsilon r. \tag{1}$$

This equation is of fundamental importance in the theory of locally isotropic turbulence (Batchelor 1947; Monin 1959; Monin & Yaglom 1975; Frisch 1995). Kolmogorov's equation (1) relates structure functions of second order $D_{11}(r)$ and third order $D_{111}(r)$ at spacing $r = |\boldsymbol{r}|$, where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{x}'$, $\boldsymbol{x}$ and $\boldsymbol{x}'$ are two position vectors, ν is kinematic viscosity, and ϵ is energy dissipation rate per unit mass of fluid. For inertial-range spacings, (1) reduces to

$$D_{111}(r) = -\tfrac{4}{5}\epsilon r, \tag{2}$$

which is known as Kolmogorov's 4/5 law, and is of practical utility for determining the energy dissipation rate. Two incompressibility relationships were used to obtain (1); these are

$$\frac{r}{2}\frac{\mathrm{d}D_{11}(r)}{\mathrm{d}r}+D_{11}(r)-D_{\gamma\gamma}(r)=0 \tag{3}$$

and

$$r\frac{\mathrm{d}D_{111}(r)}{\mathrm{d}r}+D_{111}(r)-6D_{1\gamma\gamma}(r)=0. \tag{4}$$

Subscript 1 denotes a component in the direction of $\boldsymbol{r}$, and greek subscripts are 2 or 3, which denote the Cartesian components transverse to $\boldsymbol{r}$. In §6, it is found that (4) is not essential to obtain (1). Recently, the basis of (1) and (4) (i.e. the assumptions used in their derivation) has been the subject of constructive criticism (Nelkin 1994; Frisch 1995; Lindborg 1996). The resolution of these criticisms is one subject of this paper. The derivation of equations valid for local homogeneity without use of local isotropy is also a subject of this paper, as is the investigation of the applicability of the equations obtained both with and without local isotropy.

For clarity and brevity, it is useful to give some definitions before introducing the criticisms. The nomenclature of Monin & Yaglom (1975) is used with regard to isotropy versus local isotropy, homogeneity versus local homogeneity, and stationarity versus local stationarity. In particular, isotropy and homogeneity mean what others sometimes refer to as *global isotropy* and *global homogeneity*, respectively. For brevity, the Navier–Stokes equation, incompressibility, local homogeneity, and local isotropy are collectively referred to as *the postulates*. A derivative with respect to the x_i-component of the position vector $\boldsymbol{x}$ is denoted by ∂_i, and a derivative with respect to the x_i'-component of $\boldsymbol{x}'$ is denoted by ∂_i'. Repeated roman subscripts imply summation, but no summation is implied by repeated greek subscripts. The velocity vector $\boldsymbol{u}$ satisfies the Navier–Stokes equation, and the fluid is incompressible such that $\boldsymbol{u}$ is solenoidal (i.e. $\partial_i u_i = 0$). In contrast, $\boldsymbol{V}$ denotes a general, solenoidal ($\partial_i V_i = 0$), random vector field such that $\boldsymbol{V}$ need not satisfy the Navier–Stokes equation. What is true for $\boldsymbol{V}$ is true for $\boldsymbol{u}$. Following Monin & Yaglom (1975), relationships will be demonstrated for statistics of $\boldsymbol{V}$ in preference to statistics of $\boldsymbol{u}$, whenever possible. This has the advantage of emphasizing that those relationships do not depend on the Navier–Stokes equation, as well as the advantage of generality. To produce general results, derivations are carried out using local homogeneity to the furthest extent possible; the additional restriction of local isotropy is introduced only when necessary. Generality is motivated by the fact that it obviates incrementally more general derivations. Relationships derived for statistics of $\boldsymbol{V}$ are not solely kinematic because, for instance, incompressibility influences the dynamics of $\boldsymbol{u}$ in an essential way. Statistics of $\boldsymbol{V}$ will be distinguished from those of $\boldsymbol{u}$ by including $\boldsymbol{V}$ in the argument list, e.g. $D_{ij}(\boldsymbol{V};\boldsymbol{r})$.

Nelkin (1994) referred to Frisch (1995) as showing that Kolmogorov's equation has been demonstrated only on the basis of isotropy. According to Nelkin (1994), proving Kolmogorov's equation on the basis of local isotropy remains a theoretical challenge. With reference to the book by Monin & Yaglom (1975), Frisch (1995, §6.2.5) identified the problem as being that (4) has been proven only for the isotropic case. Below their equation (13.91), Monin & Yaglom (1975) outlined a method for proof of (4) for $\boldsymbol{V}$, but no proof is given, and they described the proof as being very complicated. In §5, a relatively simple proof of (4) for $\boldsymbol{V}$ is given. In fact, Monin (1959) derived (4) for the locally isotropic case of the velocity field $\boldsymbol{u}$, although his derivation requires a relationship that he did not prove. Monin's (1959) derivation of (4) for $\boldsymbol{u}$ is terse and is even more so below equation (22.15) of Monin & Yaglom (1975). Therefore, his derivation of (4) is expounded in §4. Lindborg's (1996) criticism is best stated after a brief review of Monin's (1959) procedure for deriving (1).

Monin (1959) sought to establish (1) and (4) on the basis of local isotropy. From the Navier–Stokes equation, he obtained an equation for the two-point difference of velocities and used that equation to obtain Kolmogorov's equation (1). Following Monin's procedure, one obtains

$$\dot{D}_{ij}(\boldsymbol{r})+D_{ijk}(\boldsymbol{r})_{|k}=-T_{ij}(\boldsymbol{r})+2\nu D_{ij}(\boldsymbol{r})_{|kk}-E_{ij}(\boldsymbol{r}), \tag{5}$$

where

$$D_{ij}(\boldsymbol{r})\equiv\langle(u_i-u_i')(u_j-u_j')\rangle, \tag{6a}$$

$$D_{ijk}(\boldsymbol{r})\equiv\langle(u_i-u_i')(u_j-u_j')(u_k-u_k')\rangle, \tag{6b}$$

$$T_{ij}(\boldsymbol{r})\equiv\frac{1}{\rho}[\langle(u_i-u_i')(\partial_j P-\partial_j' P')\rangle+\langle(u_j-u_j')(\partial_i P-\partial_i' P')\rangle], \tag{6c}$$

$$E_{ij}(\boldsymbol{r})\equiv 2\nu\langle(\partial_k u_i)(\partial_k u_j)+(\partial_k' u_i')(\partial_k' u_j')\rangle, \tag{6d}$$

u_i and u_i' are evaluated at $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively, as are pressures P and P', and ρ is density. In (5), the dot above $D_{ij}(\boldsymbol{r})$ denotes differentiation with respect to time. The subscript vertical bar followed by indices denotes differentiation with respect to components of $\boldsymbol{r}$; for example, $|k$ means $\partial/\partial r_k$ and $|kk$ is the Laplacian operator. Angle brackets denote averages. Following the derivation by Monin (1959), local homogeneity has been used to commute the derivatives from within to outside the averages for the terms $D_{ijk}(\boldsymbol{r})_{|k}$ and $2\nu D_{ij}(\boldsymbol{r})_{|kk}$ in (5). To obtain (5), no assumptions are needed for the other terms. Monin eliminated the time-derivative term because he considered stationary turbulence. He eliminated the term $T_{ij}(\boldsymbol{r})$ on the basis that velocity differences are not correlated with pressure differences for locally isotropic turbulence. Without those two terms, (5) is the same as Monin's equation (7). For local isotropy, $E_{ij}=a\delta_{ij}$, and a can be evaluated by setting $i=j=1$ or 2 or 3 or summing these three index values; thus,

$$\begin{aligned} a &= 4\nu\langle(\partial_k u_1)(\partial_k u_1)\rangle = 4\nu\langle(\partial_k u_\gamma)(\partial_k u_\gamma)\rangle \\ &= \frac{4\nu}{3}\langle(\partial_k u_j)(\partial_k u_j)\rangle = \tfrac{4}{3}[\epsilon-\nu\langle(\partial_i u_j)(\partial_j u_i)\rangle]. \end{aligned} \tag{7}$$

Monin showed that $\langle(\partial_i u_j)(\partial_j u_i)\rangle$ is zero on the basis of local isotropy; in §5, it is found to vanish on the basis of local homogeneity without use of local isotropy. Combining these results, (5) becomes

$$\dot{D}_{ij}(\boldsymbol{r})+D_{ijk}(\boldsymbol{r})_{|k}=2\nu D_{ij}(\boldsymbol{r})_{|kk}-\tfrac{4}{3}\epsilon\delta_{ij}. \tag{8}$$

Local isotropy was used to simplify only the right-most term in (8). Excluding the time-derivative term, (8) is Monin's equation (8).

Lindborg (1996) showed that Monin's reasons for neglecting $T_{ij}(\boldsymbol{r})$ are incorrect. In his Appendix, Lindborg proved it impossible that Monin's (1959) reasons have anything to do with neglect of $T_{ij}(\boldsymbol{r})$. As shown in §3, $T_{ij}(\boldsymbol{r})$ does vanish on the basis of local homogeneity, local isotropy, and incompressibility. Monin's equation (8) is thus proven on the basis of the postulates. Kolmogorov's equation (1) is derived from Monin's equation (8) by Monin (1959), as well as Monin & Yaglom (1975). They used the isotropic formulae

$$D_{ij}(\boldsymbol{r})=[D_{11}(r)-D_{\gamma\gamma}(r)]\frac{r_i r_j}{r^3}+D_{\gamma\gamma}(r)\,\delta_{ij} \tag{9}$$

and

$$D_{ijk}(\boldsymbol{r})=[D_{111}(r)-3D_{1\gamma\gamma}(r)]\frac{r_i r_j r_k}{r^3}+D_{1\gamma\gamma}(r)\left(\frac{r_i}{r}\delta_{jk}+\frac{r_j}{r}\delta_{ik}+\frac{r_k}{r}\delta_{ij}\right). \tag{10}$$

They used (3) and (4), performed two integrations with respect to r, and used the fact that expressions in their procedure are regular at $r = 0$. Considering results in the present paper, their procedure uses only formulae that are valid under local isotropy. Kolmogorov's equation (1) is thus proven on the basis of the postulates and local stationarity.

Kolmogorov (1941) included local stationarity of temporal increments in his definition of local homogeneity. Following Monin & Yaglom (1975), local homogeneity means homogeneity of spatial increments. Again following Monin & Yaglom (1975), one can take ϵ to be of the order of the cube of the root-mean-squared velocity divided by an external scale L, and the time scale for changes in ϵ to be of the order of the external scale divided by the root-mean-squared velocity. Then, for the inertial range, the time-derivative term in (8) is of order $(r/L)^{2/3}$ times the term proportional to ϵ. Thus, the time-derivative term is negligible at sufficiently small spacings and large Reynolds numbers, i.e. for $r \ll L$. In other words, the limit of small spacings and large Reynolds numbers is also the locally stationary limit. A discussion of this fact was given by Batchelor (1947).

On the other hand, although it may be a very important case, the locally stationary case is not the only case for which these equations apply. For moderate Reynolds numbers, the approximation $r \ll L$ might not be sufficiently accurate for neglect of the time-derivation term in some cases, whereas (5) applies if the turbulence is nearly homogeneous and (7) applies for nearly isotropic turbulence. For high-Reynolds-number geophysical turbulence, non-stationarity is only controlled by data selection. In this case, the time-derivative term can be relevant to estimating how large r can be, and yet have inertial-range laws that are approximately obeyed. For the isotropic case, (1) reduces to (2) for arbitrarily large r, showing that (1) applies only for infinite Reynolds numbers for the isotropic case. However, for finite Reynolds numbers, returning the time-derivative term to (1) allows the time-derivative term to balance the right-hand side of (1) such that $D_{111}(r)$ approaches zero, as it should for $r \to \infty$. The time-derivative term will be retained in subsequent sections because it allows an increase in generality with no increase in complexity.

It is important to describe experimental conditions covered by the various derivations of Kolmogorov's equation (1). Kolmogorov (1941) did not give a derivation of (1): he only referred to it as being similar to the Kármán–Howarth equation (Kármán & Howarth 1938). Batchelor (1947) and Monin & Yaglom (1975), among others, gave the derivation of (1) from the Kármán–Howarth equation. In so doing, the energy dissipation rate is introduced from the time rate of change of velocity variance, which, as emphasized by Frisch (1995) and Lindborg (1996), makes the derivation apply to freely decaying turbulence. Monin's equation (8) is different. Having retained the time-derivative term in (8), the equation applies whether the turbulence is stationary, decaying, or increasing in energy, or of high enough Reynolds number to be locally stationary for small enough r independent of the rate of decay or increase of energy. There are no forces introduced in Monin's derivation to produce turbulent energy (or to damp it); this corresponds to the typical experimental situation of forces being exerted at positions other than the points of measurement $\boldsymbol{x}$ and $\boldsymbol{x}'$. Examples are turbulence generated by a grid and measured downstream, as well as measurements in the boundary layer of a wall. Therefore, Monin's derivation is of practical applicability. Frisch (1995) gave a derivation using a homogeneous force; a practical example is turbulence that is driven by nearly homogeneous buoyancy. It seems that Kolmogorov understood the generality of his equation, whatever the method of derivation.

Kolmogorov's 4/5 law (2) is derived in §6 by use of local isotropy, but without use of (3) and (4). Therefore, for any given data, accuracy of (3) or (4) is not a necessary condition for the data to exhibit the 4/5 law. This can be of some practical relevance to experimentalists because our experience with nearly homogeneous wind-tunnel grid turbulence suggests that (4) is not easily satisfied even when (3) is satisfied. Furthermore, generalizations of Kolmogorov's 4/5 law are given by Lindborg (1996) and Frisch (1995). The basis of their derivations includes homogeneity and incompressibility. In §5, local homogeneity is used to obtain their generalization.

The operational calculus of local homogeneity is used in the following sections. The formal description of local homogeneity is given by Monin & Yaglom (1975), so a brief introduction suffices here. Consider any statistic containing at least one difference of a turbulence quantity, e.g. $(P-P')$ or (u_i-u_i'), or containing at least one derivative. The fact that local homogeneity applies to statistics of derivatives is discussed by Monin & Yaglom (1975, end of §13.2). Local homogeneity means that such a statistic is very rapidly varying with respect to $\boldsymbol{r}$ relative to its variation with respect to $\boldsymbol{X} \equiv (\boldsymbol{x}+\boldsymbol{x}')/2$, provided that r is sufficiently small. On this basis, when derivatives with respect to $\boldsymbol{x}$ or $\boldsymbol{x}'$ are transformed into derivatives with respect to $\boldsymbol{r}$ and $\boldsymbol{X}$, the differentiation with respect to $\boldsymbol{X}$ is negligible compared to that with respect to $\boldsymbol{r}$. The operational calculus of local homogeneity is that derivatives with respect to x_i (and x_k') within the averaging operation can be commuted to outside the average such that they become derivatives with respect to r_i (and $-r_k$, in the case of x_k'); also, derivatives with respect to r_i can be commuted to inside the average where they may be performed with respect to either x_i or $-x_i'$.

2. Approach toward local isotropy

In §3, it is shown that $T_{ij}(\boldsymbol{r})$ vanishes on the basis of local isotropy. Here, the way in which local isotropy is approached in anisotropic turbulence is considered. Emphasis is placed on determining the restrictions needed in passing from (5) to (8).

Consider nearly homogeneous turbulence of a moderate Reynolds number. If local homogeneity is adequate at dissipation scales, then local homogeneity is adequate for the term $2\nu D_{ij}(\boldsymbol{r})_{|kk}$ in (5) because this term is significant only at dissipation-range scales. The only other term in (5) that was approximated using local homogeneity is $D_{ijk}(\boldsymbol{r})_{|k}$; this term is significant at larger scales and is therefore the most relevant term with regard to the accuracy of (5). For nearly isotropic turbulence at moderate Reynolds numbers, (5) applies, and the locally isotropic approximation $E_{ij} = (4\epsilon/3)\,\delta_{ij}$ is accurate because of the dominance of small scales in the derivative moment (6*d*). At this point, passing from (5) to (8) only requires eliminating the term $T_{ij}(\boldsymbol{r})$ on the basis of local isotropy.

The approach to local isotropy with decreasing r has been studied in many ways. Significant recent studies are the experiment by Saddoughi & Veeravalli (1994) and the numerical simulations by Borue & Orszag (1996), Pumir & Shraiman (1995), and Pumir (1996). The most typical statistics studied for the approach to local isotropy are the ratios of the off-diagonal components of the velocity spectrum tensor to its on-diagonal components. These statistics are studied by Saddoughi & Veeravalli (1994) and Borue & Orszag (1996), as well as by others. In terms of structure functions, the result is that for $\alpha \neq \beta$, $D_{\alpha\beta}(r)/D_{11}(r)$ decreases proportional to $r^{2/3}$ in the inertial range. Recall from the previous discussion that local stationarity reduces the importance of $\dot{D}_{ij}(\boldsymbol{r})$ as r decreases and that the terms $2\nu D_{ij}(\boldsymbol{r})_{|kk}$ and $E_{ij}(\boldsymbol{r})$ in (5) are most influenced by dissipation-range scales for which local isotropy is at least as good an approximation as for larger scales. For these reasons, the approach to local isotropy of $D_{ij}(\boldsymbol{r})$ is not the most relevant question with regard to the accuracy of passing from (5) to (8). More relevant is the approach to zero of $T_{ij}(\boldsymbol{r})$, which, from (5), must be approximately balanced by the approach of $D_{ijk}(\boldsymbol{r})_{|k}$ to its isotropic form as r decreases in the inertial range; this point is considered further with regard to (39) in §5.

Borue & Orszag (1996) have performed the relevant study by calculating the cross-spectrum of the velocity and the pressure gradient. In terms of $T_{ij}(\boldsymbol{r})$, their result is that in the inertial range $T_{ij}(\boldsymbol{r})/\epsilon$ decreases proportional to r. Here, ϵ is used to scale $T_{ij}(\boldsymbol{r})$ because the magnitudes of the other important inertial-range terms in (5) are approximately $4\epsilon/3$, which is, in turn, approximately equal to the diagonal components of $E_{ij}(\boldsymbol{r})$. In the inertial range, the anisotropic contribution to (5) of $D_{ijk}(\boldsymbol{r})_{|k}$ must also decrease proportional to r such that (5) balances. Use in (8) of the isotropic formula (10) for $D_{ijk}(\boldsymbol{r})$ is therefore as strong an approximation as is neglect of $T_{ij}(\boldsymbol{r})$, but use of the isotropic formula (9) for $D_{ij}(\boldsymbol{r})$ is a relatively weaker approximation. Thus, the accuracy of Kolmogorov's equation (1) is more sensitive to accuracy of the third-order isotropy–incompressibility relationship (4) than to accuracy of the second-order isotropy–incompressibility relationship (3). Since use of the isotropic formula (10) for $D_{ijk}(\boldsymbol{r})$ is as strong an approximation as neglect of $T_{ij}(\boldsymbol{r})$ in passing from (5) to (8), it follows that (8) should be considered as an intermediate step that is no more general than considerably simplified equations resulting from substitution of (9) and (10) in (8); those simplified equations are (41) and (42) in §6.

3. $T_{ij}(\boldsymbol{r})$ vanishes on the basis of local isotropy

Here, we show on the basis of local homogeneity, local isotropy, and incompressibility that $T_{ij}(\boldsymbol{r})$ vanishes and, therefore, that Monin's equation (8) is proven on the basis of the postulates. Thus, Lindborg's (1996) criticism is resolved. If, in (6*c*), velocity $\boldsymbol{u}$ is replaced by $\boldsymbol{V}$ and pressure P is replaced by an arbitrary scalar S, then the following derivation remains valid: S could be a dynamically passive additive or dynamically active, just as P is dynamically active. That is, the following derivation applies to the general statistic:

$$T_{ij}(\boldsymbol{V}, S; \boldsymbol{r}) = \langle (V_i - V_i')(\partial_j S - \partial_j' S') \rangle + (i \leftrightarrow j), \tag{11}$$

where the notation $+(i \leftrightarrow j)$ means add the expression given explicitly with i and j interchanged. The derivation will be carried out in the notation $\boldsymbol{u}$ and P. Use is made of the facts that $\partial_i u_i = \partial_i' u_i' = 0$ and that $\partial_j' u_i$, $\partial_i u_i'$, $\partial_j' P$, and $\partial_j P'$ are all zero because x_j' is held fixed for the partial derivatives with respect to x_j and vice versa.

First, on the basis of local homogeneity, (6*c*) is expressed as the pressure–strain-rate structure function, from which it follows immediately that $T_{ii}(\boldsymbol{r}) = 0$; equation (6*c*) is written as

$$T_{ij}(\boldsymbol{r}) = \frac{1}{\rho}[\langle (u_i - u_i')\,\partial_j (P - P') \rangle + \langle (u_i - u_i')\,\partial_j'(P - P') \rangle] + (i \leftrightarrow j). \tag{12}$$

Now,

$$\begin{aligned}\langle (u_i - u_i')\,\partial_j (P - P') \rangle &= \langle \partial_j [(u_i - u_i')(P - P')] - (\partial_j u_i)(P - P') \rangle \\ &= \langle (u_i - u_i')(P - P') \rangle_{|j} - \langle (\partial_j u_i)(P - P') \rangle \qquad (13) \\ &= -\langle \partial_j' [(u_i - u_i')(P - P')] \rangle - \langle (\partial_j u_i)(P - P') \rangle \qquad (14) \\ &= -\langle (u_i - u_i')\,\partial_j'(P - P') \rangle + \langle (\partial_j' u_i')(P - P') \rangle - \langle (\partial_j u_i)(P - P') \rangle. \qquad (15)\end{aligned}$$

Steps (13) and (14) require local homogeneity. Substituting (15) for the first term in (12) gives

$$T_{ij}(\boldsymbol{r}) = \frac{1}{\rho}\langle(\partial'_j u'_i - \partial_j u_i)(P-P')\rangle + (i \leftrightarrow j). \tag{16}$$

Summing over indices in (16) produces $(\partial'_i u'_i - \partial_i u_i)$, which vanishes. Thus, on the basis of incompressibility and local homogeneity,

$$T_{ii}(\boldsymbol{r}) = 0. \tag{17}$$

The divergence on index i of the first term in (6*c*) is

$$\frac{1}{\rho}\langle(u_i - u'_i)(\partial_j P - \partial'_j P')\rangle_{|i} = \frac{1}{\rho}\langle(u_i - u'_i)\,\partial_i\partial_j P\rangle = \frac{1}{\rho}\langle(u_i - u'_i)\,\partial_j P\rangle_{|i}$$

$$= -\frac{1}{\rho}\langle\partial'_i(u_i - u'_i)\,\partial_j P\rangle = 0. \tag{18}$$

By symmetry of $T_{ij}(\boldsymbol{r})$, the divergence on the index j of the second term in (6*c*) also vanishes. Therefore, on the basis of local homogeneity and incompressibility,

$$T_{ij}(\boldsymbol{r})_{|ij} = 0. \tag{19}$$

As shown in the Appendix, the curl operating twice on $T_{ij}(\boldsymbol{r})$ also vanishes.

Applying (19) to the isotropic formula (9), as applied to $T_{ij}(\boldsymbol{r})$, gives

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\left\{\frac{\mathrm{d}T_{11}(r)}{\mathrm{d}r} + \frac{2}{r}[T_{11}(r) - T_{\gamma\gamma}(r)]\right\}\right) = 0, \tag{20}$$

which has the solution, in terms of the constant of integration b,

$$\frac{r}{2}\frac{\mathrm{d}T_{11}(r)}{\mathrm{d}r} + T_{11}(r) - T_{\gamma\gamma}(r) = \frac{b}{r} = 0, \tag{21}$$

wherein $b = 0$ because Taylor series expansion of (6*c*) shows that the left-hand side of (21) is zero at $r = 0$. Finally, condition (17) combined with (9), as applied to $T_{ij}(\boldsymbol{r})$, gives $2T_{\gamma\gamma}(r) = T_{11}(r)$, substitution of which into (21) gives

$$\frac{\mathrm{d}}{\mathrm{d}r}[rT_{11}(r)] = 0. \tag{22}$$

Solution of (22) in terms of the constant of integration c gives $T_{11}(r) = c/r$, but $T_{11}(r)$ is regular at $r = 0$, so $c = 0$. Thus, $T_{11}(r)$ and $T_{\gamma\gamma}(r)$ are zero. That is, on the basis of local isotropy, local homogeneity, and incompressibility,

$$T_{ij}(\boldsymbol{r}) = 0. \tag{23}$$

Finally, some notes are worthwhile on alternative derivations that use local isotropy in steps prior to (20). To derive (16), and therefore (17), on the more restrictive basis of local isotropy, it is sufficient to use (13) because, as shown in §13.3 of Monin &

Yaglom (1975), the first average in (13) vanishes on the basis of local isotropy. As noted, the divergence on the index j of the second term in (6c) vanishes. Thus, the divergence on index i of this second term is a vector having vanishing divergence on its sole index j. On the basis of local isotropy, every such solenoidal vector vanishes (Monin & Yaglom 1975). This fact, combined with (18), gives

$$T_{ij}(\boldsymbol{r})_{|i} = T_{ij}(\boldsymbol{r})_{|j} = 0. \tag{24}$$

That is, on the basis of local homogeneity and incompressibility, we have (19), and adding the more restrictive basis of local isotropy, we have (24). Of course, (21) follows from (24) and the final result, (23), can therefore follow from (24). Using (24) instead of (19) avoids the intermediate step (20), but adds the step of determining that the divergence on the index i of the second term in (6c) vanishes.

4. Monin's derivation of (4)

In their §13.3 on locally isotropic fields, Monin & Yaglom (1975) proved that for any random solenoidal vector field $\boldsymbol{V}$,

$$D_{ij}(\boldsymbol{V};\boldsymbol{r})_{|i} = D_{ij}(\boldsymbol{V};\boldsymbol{r})_{|j} = 0, \tag{25}$$

and that (9) and (25) lead to (3) on the basis of local isotropy. The proof of (25) by Monin & Yaglom (1975) is valid under local homogeneity without local isotropy, although they did not note that fact. A simpler proof of (25) is given in the next section.

Monin's (1959) method to prove (4) for $\boldsymbol{u}$ on the basis of local homogeneity and local isotropy was to perform the divergence of (8) and use (25) to obtain

$$D_{ijk}(\boldsymbol{r})_{|jk} = 0. \tag{26}$$

Substituting (10) in (26) gives

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}\right)\left\{\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}\right)[D_{111}(r)-3D_{1\gamma\gamma}(r)]\right\}+3\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{r}\right)D_{1\gamma\gamma}(r)\right]=0. \tag{27}$$

Defining $F(r)$ to be the left-hand side of (4), (27) becomes $(\mathrm{d}/\mathrm{d}r+2/r)[F(r)/r]=0$, which has the solution, in terms of the constant of integration $e, F(r) = e/r$, but $e = 0$ because the left-hand side of (4) is regular at $r = 0$. Therefore, (4) is proven on the basis of the postulates.

Frisch's (1995) criticism that (4) had been proven only on the basis of isotropy remains valid because of Lindborg's (1996) valid criticism that the justification stated by Monin (1959) was not adequate to prove that $T_{ij}(\boldsymbol{r})$ vanishes. Without such proof, $T_{ij}(\boldsymbol{r})_{|j}$ would appear on the right-hand side of (26).

One can perform the divergence of (5) and use (24) and (25) to return to (26) on the basis of the postulates. Alternatively, one can perform the double divergence of (5) and use (19) and (25) to obtain, without use of local isotropy, that

$$D_{ijk}(\boldsymbol{r})_{|ijk} = 0. \tag{28}$$

Now, (28) is relevant to deducing restrictions on $D_{ijk}(\boldsymbol{r})$ for any local symmetry property, e.g. for turbulence that is axisymmetric.

In the case of local isotropy, (10) and (28) give (27) with an additional operator $(\mathrm{d}/\mathrm{d}r+2/r)$ on the entire left-hand side of (27), but the solution remains $F(r) = 0$ because the Taylor series expansions $D_{111}(r)$ and $D_{1\gamma\gamma}(r)$ require that the additional constant of integration vanishes. Once again, (4) follows on the basis of the postulates.

5. Results for anisotropy and local homogeneity

The Navier–Stokes equation is not needed to establish (4), (27), or (28). Those results are valid for any solenoidal vector $\boldsymbol{V}$. Consider the triple divergence of the third-order structure function of $\boldsymbol{V}$:

$$D_{ijk}(\boldsymbol{V};\boldsymbol{r})_{|ijk} = \langle (V_i - V_i')(V_j - V_j')(V_k - V_k')\rangle_{|ijk} \tag{29a}$$

$$= -\langle \partial_k[\partial_i' \partial_j (V_i V_j V_k - V_i V_j V_k' - V_i V_j' V_k - V_i' V_j V_k + V_i V_j' V_k' + V_i' V_j V_k' + V_i' V_j' V_k - V_i' V_j' V_k')]\rangle \tag{29b}$$

$$= -\langle \partial_k[\partial_i' \partial_j (-V_i V_j V_k' - V_i V_j' V_k + V_i V_j' V_k' + V_i' V_j' V_k)]\rangle \tag{29c}$$

$$= -\langle \partial_i[\partial_k \partial_j' (-V_i V_j V_k' + V_i V_j' V_k')]\rangle \tag{29d}$$

$$= \langle \partial_j'[\partial_k' \partial_i (-V_i V_j V_k' + V_i V_j' V_k')]\rangle \tag{29e}$$

$$= 0. \tag{29f}$$

In passing from (29*a*) to (29*b*), the derivatives have been commuted to inside the average on the basis of local homogeneity, and the distributive law of multiplication has been applied to the product of the differences. In passing from (29*b*) to (29*c*), those terms that vanish when operated on by $\partial_i' \partial_j$ have been removed. In passing from (29*c*) to (29*d*), four steps were performed: ∂_j was commuted to outside the average to become $|j$ and commuted back inside to become $-\partial_j'$; the same commutation was subsequently performed for changing ∂_i' to ∂_i; the order of derivatives was changed; and those terms appearing in (29*c*) that vanish when operated on by $\partial_k \partial_j'$ were eliminated. Similar commutation and reordering of the derivatives give (29*e*) from (29*d*). The two terms in (29*e*) vanish when operated on by $\partial_k' \partial_i$, thereby producing (29*f*). The averages in (29*b*–*e*) vary much more rapidly with $\boldsymbol{r}$ than with $\boldsymbol{X}$ because of two facts: that the quantities in parentheses vanish as $\boldsymbol{r} \to 0$ and that the third-order derivative operation causes dissipation-range scales to dominate the averages. The rapid variation with $\boldsymbol{r}$, as compared to the variation with $\boldsymbol{X}$, allows commutation of the derivatives on the basis of local homogeneity.

The vanishing of multiple-order divergences under local homogeneity does not extend to higher orders. An important practical case is

$$D_{ijkl}(\boldsymbol{V};\boldsymbol{r})_{|ijkl} \equiv \langle (V_i - V_i')(V_j - V_j')(V_k - V_k')(V_l - V_l')\rangle_{|ijkl}$$

$$= \langle \partial_i \partial_j \partial_k' \partial_l' (V_i' V_j' V_k V_l + V_i V_j' V_k' V_l + V_i V_j' V_k V_l' + V_i' V_j V_k' V_l + V_i' V_j V_k V_l' + V_i V_j V_k' V_l')\rangle, \tag{30}$$

where the steps are to introduce $|i$ to within the average first as $-\partial_i'$, then as ∂_i, and similarly for $|j$; then $|k$ as ∂_k, then as $-\partial_k'$, and similarly for $|l$; and then eliminate those fourth-order products that vanish under the several combinations of derivatives. Now,

$$\langle \partial_i \partial_j \partial_k' \partial_l' (V_i V_j V_k' V_l')\rangle = \langle \partial_i' \partial_j \partial_k \partial_l' (V_i V_j V_k' V_l')\rangle \tag{31a}$$

$$= \langle \partial_i' \partial_j' \partial_k \partial_l (V_i V_j V_k' V_l')\rangle, \tag{31b}$$

where (31*a*) follows by removing ∂_i, reintroducing $|i$ as $-\partial_i'$, subsequently removing $-\partial_k'$, and reintroducing $|k$ as ∂_k; (31*b*) follows from (31*a*) using analogous operations on ∂_j and ∂_l'. Use of (31*a*, *b*) shows that all six terms within (30) are equal; relabelling of indices is used for some of the terms. The fourth-order differentiation in (30) and (31 *a*, *b*) produces much more rapid variation of the average with $\boldsymbol{r}$ than with $\boldsymbol{X}$, thereby allowing commutation of one derivative at a time on the basis of local homogeneity. Therefore, on the basis of local homogeneity alone, substitution of (31 *a*, *b*) in (30) gives

$$D_{ijkl}(\boldsymbol{V};\boldsymbol{r})_{|ijkl} = 6\langle \partial_j V_i \partial_i V_j \partial'_l V'_k \partial'_k V'_l \rangle. \tag{32}$$

The derivative moment on the right-hand side of (32), when applied to velocity $\boldsymbol{u}$, is the quantity that leads to the pressure structure function and spectrum (Obukhov 1949; Batchelor 1951). The relationship (32) to the fourth-order divergence of $D_{ijkl}(\boldsymbol{r})$ was used by Hill & Wilczak (1995) to obtain results for the pressure structure function and spectrum that are valid solely on the basis of the postulates. The derivation of (32) given above is much simpler than the derivation given by Hill & Wilczak (1995) and is more general than the derivation given by Hill (1993).

On the basis of local homogeneity alone, steps analogous to (29 *a–f*) give (25) as follows:

$$\begin{aligned} D_{ij}(\boldsymbol{V};\boldsymbol{r})_{|i} &= \langle (V_i - V'_i)(V_j - V'_j)\rangle_{|i} \\ &= \langle (V_i - V'_i)(\partial_i V_j)\rangle = \langle (V_i - V'_i) V_j\rangle_{|i} \\ &= -\langle \partial'_i [(V_i - V'_i) V_j]\rangle = 0. \end{aligned} \tag{33}$$

This is a simpler proof of (25) than given by Monin & Yaglom (1975). Now, (33) requires $D_{ij}(\boldsymbol{V};\boldsymbol{r})_{|ij} = 0$, so

$$D_{ij}(\boldsymbol{V};\boldsymbol{r})_{|ij} = \langle \partial_j (V_i - V'_i)\partial_i (V_j - V'_j)\rangle = \langle (\partial_j V_i)(\partial_i V_j)\rangle = 0. \tag{34}$$

Use of (34) shows that

$$\langle (\partial_k V_i)(\partial_k V_i)\rangle = \tfrac{1}{2}\langle (\partial_i V_j + \partial_j V_i)(\partial_i V_j + \partial_j V_i)\rangle. \tag{35}$$

Use of (34) also shows that the derivative moment in (7), namely $\langle (\partial_i u_j)(\partial_j u_i)\rangle$, vanishes on the basis of local homogeneity and incompressibility. For $\boldsymbol{u}$, this derivative moment is also, on the basis of the Navier–Stokes equation and incompressibility, the average of the Laplacian of pressure, which also clearly vanishes by local homogeneity.

Use of (35) in the definition of $E_{ij}(\boldsymbol{r})$ in (6 *d*) shows that local homogeneity requires

$$E_{ii}(\boldsymbol{r}) = 4\epsilon. \tag{36}$$

Performing the trace of (5), and using (17) and (36), gives

$$\dot{D}_{ii}(\boldsymbol{r}) + D_{iik}(\boldsymbol{r})_{|k} = 2\nu D_{ii}(\boldsymbol{r})_{|kk} - 4\epsilon. \tag{37}$$

For locally stationary turbulence (so either $r \ll L$ or the turbulence is stationary) and for r very much greater than Kolmogorov's microscale such that the term $2\nu D_{ii}(\boldsymbol{r})_{|kk}$ can be neglected, (37) is approximately

$$D_{iik}(\boldsymbol{r})_{|k} \approx -4\epsilon. \tag{38}$$

Both (37) and (38) are obtained on the basis of local homogeneity without use of local isotropy. As shown in the Appendix, the curl operating twice on (5) can, in some cases, give conditions on $D_{ijk}(\boldsymbol{r})$ that are in addition to (38). Now, (38) is essentially the generalization of Kolmogorov's 4/5 law obtained by Lindborg (1996) and Frisch (1995), both of whom assumed homogeneity. For homogeneous turbulence, there is no restriction on r for the accuracy of the right-hand side of (37), and for stationary turbulence, the time-derivative term in (37) vanishes by definition (without the restriction $r \ll L$). In this case, (38) follows from (37) only on the basis of the additional restriction that r be very much greater than Kolmogorov's microscale. That is, in addition to the applicability of (38) to the inertial range, (38) applies to the energy-containing range of homogeneous, stationary turbulence (even for moderate Reynolds numbers such that $2\nu D_{ii}(\boldsymbol{r})_{|kk}$ can be neglected). Of course, validity of (37) and (38) in this case requires that there be no external forces at points $\boldsymbol{x}$ and $\boldsymbol{x}'$. Since (38) applies to a stationary and homogeneous shear flow, deviations from (38) in a real shear flow are a measure of inhomogeneity of the flow (to the extent that the viscous term is negligible). Study of (37) and (38) for simulated homogeneous turbulence can shed further light on the persistence of local anisotropy found by Pumir (1996).

Performing the divergence of (5) and using (25),

$$D_{ijk}(\boldsymbol{r})_{|jk} = -T_{ij}(\boldsymbol{r})_{|j}, \tag{39}$$

which is valid on the basis of local homogeneity without invoking local isotropy. Of course, both sides of (39) vanish on the basis of local isotropy because of (24) and (26). Axisymmetry is the next simplest symmetry after isotropy (Batchelor 1946; Lindborg 1996). Substitution of formulae for axisymmetric tensors in (39) gives relationships that on the left-hand side of (39) can be evaluated using velocity measurements and thereby constrain the approach to zero of $T_{ij}(\boldsymbol{r})$ as local isotropy is approached for decreasing r and increasing Reynolds numbers. Although such velocity measurements require that several velocity components be measured for separations both along and transverse to the axis of symmetry, pressure measurements are not needed. Thus, (39) is a starting point for experimental evaluation of $T_{ij}(\boldsymbol{r})$ for anisotropic, locally homogeneous turbulence.

6. Results for local homogeneity and local isotropy

Returning to (29*c*), but removing ∂_k and further commuting the derivatives to produce $\partial_i \partial'_j$ from $\partial'_i \partial_j$ gives

$$D_{ijk}(V;\boldsymbol{r})_{|ij} = \langle \partial_i \partial'_j(-V_i V_j V'_k + V'_i V'_j V_k)\rangle, \tag{40a}$$

$$= 0, \tag{40b}$$

where (40*b*) follows from (40*a*) on the basis of local isotropy. Thus, local homogeneity and local isotropy give (26) for $\boldsymbol{u}$ and give (40*b*) for V independent of the Navier–Stokes equation. Recall that the Navier–Stokes equation was used by Monin (1959) to obtain (26). Of course, (4) for V follows from (40*b*) by means of the derivation in §4.

Next, derivation of Kolmogorov's equations (1) and (2) is considered. Substituting (9) and (10) in (8) gives two scalar equations:

$$\dot{D}_{11}(r) + \frac{\mathrm{d}D_{111}(r)}{\mathrm{d}r} + \frac{2}{r}[D_{111}(r) - 2D_{1\gamma\gamma}(r)] = 2\nu\left\{\frac{\mathrm{d}^2 D_{11}(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}D_{11}(r)}{\mathrm{d}r} + \frac{4}{r^2}[D_{\gamma\gamma}(r) - D_{11}(r)]\right\} - \tfrac{4}{3}\epsilon, \tag{41}$$

$$\dot{D}_{\gamma\gamma}(r) + \frac{\mathrm{d}D_{1\gamma\gamma}(r)}{\mathrm{d}r} + \frac{4}{r}D_{1\gamma\gamma}(r) = 2\nu\left\{\frac{\mathrm{d}^2 D_{\gamma\gamma}(r)}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}D_{\gamma\gamma}(r)}{\mathrm{d}r} - \frac{2}{r^2}[D_{\gamma\gamma}(r) - D_{11}(r)]\right\} - \tfrac{4}{3}\epsilon. \tag{42}$$

Solution of (42) is

$$D_{1\gamma\gamma}(r) = -\tfrac{4}{15}\epsilon r + 2\nu\left\{\frac{\mathrm{d}D_{\gamma\gamma}(r)}{\mathrm{d}r} - \frac{2}{r}D_{\gamma\gamma}(r) + \frac{2}{r^4}\int_0^r \mathrm{d}y\, y^2[D_{11}(y) + 2D_{\gamma\gamma}(y)]\right\} - \frac{1}{r^4}\int_0^r \mathrm{d}y\, y^4 \dot{D}_{\gamma\gamma}(y). \quad (43)$$

Substituting (43) in (41) and solving gives

$$D_{111}(r) = -\tfrac{4}{5}\epsilon r + 2\nu\left\{\frac{\mathrm{d}D_{11}(r)}{\mathrm{d}r} + \frac{4}{r}D_{\gamma\gamma}(r) - \frac{4}{r^4}\int_0^r \mathrm{d}y\, y^2[D_{11}(y) + 2D_{\gamma\gamma}(y)]\right\} - \frac{1}{r^2}\int_0^r \mathrm{d}y\, y^2\{\dot{D}_{11}(y) + 2\dot{D}_{\gamma\gamma}(y)\,[1-(y/r)^2]\}. \quad (44)$$

Neglecting the time derivatives on the basis of local stationarity and neglecting the terms proportional to viscosity, (43) and (44) become

$$3D_{1\gamma\gamma}(r) = D_{111}(r) = -\tfrac{4}{5}\epsilon r, \quad (45)$$

which is Kolmogorov's 4/5 law without use of (3) and (4). Incompressibility and local isotropy were used in ways other than (3) and (4).

Neglect of the time-derivative term in (44) and simplification of the term proportional to viscosity by use of (3) gives Kolmogorov's equation (1) without use of (4). Thus, any given data need not accurately satisfy (4) in order that (1) and the 4/5 law might be approximately satisfied.

7. Summary

The criticisms of Kolmogorov's equation (1) have been resolved, and various relationships have been proven for the first time or proven on more general grounds than before. Tables 1–4 summarize the relationships and provide convenient lists for applications. Surprisingly, only (3), (32), and (33) are not, in any way, established on a less restricted basis than by their previous derivations.

The relationships derived here that do not require local isotropy can be used to study the approach toward local isotropy. Study of the approach to local isotropy of $D_{ijk}(\boldsymbol{r})$ is most relevant because the accuracy of Kolmogorov's equation depends more sensitively on anisotropy of $D_{ijk}(\boldsymbol{r})$ than on anisotropy of $D_{ij}(\boldsymbol{r})$. The objective of such a study would be to obtain expressions that can be tested experimentally. Experiments on nearly axisymmetric turbulence for which all relevant quantities are measured would be advantageous in this regard.

The wide range of applicability to experiments of Kolmogorov's and Monin's equations have been discussed in the Introduction. Local homogeneity combined with incompressibility, the Navier–Stokes equation, and the local isotropy formulas (9) and (10) were used to obtain (43)–(45), but (3) and (4), which require the combination of local isotropy and incompressibility, were not used. Thus, (3) and (4) are not needed to obtain Kolmogorov's 4/5 law (2) and (45); (4) is not needed to obtain his equation (1); and (3) and (4) are not needed to obtain the closely related equation (44). The generalization of the 4/5 law is (38), and (38) is applicable even to the energy-containing range, given the restrictions discussed below (38).

The statistical tensors can be expressed in the Taylor series of the variable $\boldsymbol{X} \equiv (\boldsymbol{x}+\boldsymbol{x}')/2$. From the description of the operational calculus of local homogeneity at the

$T_{ii}(\boldsymbol{V}, S; \boldsymbol{r}) = 0$	(11), (17)
$T_{ij}(\boldsymbol{V}, S; \boldsymbol{r})_{\vert ij} = 0$	(11), (19)
$D_{ij}(\boldsymbol{V}; \boldsymbol{r})_{\vert i} = D_{ij}(\boldsymbol{V}; \boldsymbol{r})_{\vert j} = 0$	(25), (33)
$D_{ijk}(\boldsymbol{V}; \boldsymbol{r})_{\vert ijk} = 0$	(29*f*)
$D_{ijkl}(\boldsymbol{V}; \boldsymbol{r})_{\vert ijkl} = 6\langle \partial_j V_i \partial_i V_j \partial'_l V'_k \partial'_k V'_l \rangle$	(32)
$\langle \partial_j V_i \partial_i V_j \rangle = 0$	(34)
$\langle (\partial_k V_i)(\partial_k V_i) \rangle = \frac{1}{2}\langle (\partial_i V_j + \partial_j V_i)(\partial_i V_j + \partial_j V_i) \rangle$	(35)

TABLE 1. Relationships proven for any locally homogeneous and solenoidal vector field $\boldsymbol{V}$ and locally homogeneous scalar field S. Relevant equation numbers are given on the right-hand side.

$T_{ij}(\boldsymbol{V}, S; \boldsymbol{r}) = 0$	(11), (23)
$D_{ijk}(\boldsymbol{V}; \boldsymbol{r})_{\vert ij} = D_{ijk}(\boldsymbol{V}; \boldsymbol{r})_{\vert ik} = D_{ijk}(\boldsymbol{V}; \boldsymbol{r})_{\vert jk} = 0$	(40*b*)
$\dfrac{r}{2}\dfrac{\mathrm{d}D_{11}(\boldsymbol{V}; r)}{\mathrm{d}r} + D_{11}(\boldsymbol{V}; r) - D_{\gamma\gamma}(\boldsymbol{V}; r) = 0$	(3), (25), (33)
$r\dfrac{\mathrm{d}D_{111}(\boldsymbol{V}; r)}{\mathrm{d}r} + D_{111}(\boldsymbol{V}; r) - 6D_{1\gamma\gamma}(\boldsymbol{V}; r) = 0$	(4), (29*f*), (40*b*)

TABLE 2. Relationships, in addition to those in table 1, proven for any locally homogeneous and locally isotropic, solenoidal vector field $\boldsymbol{V}$ and locally homogeneous and locally isotropic scalar field S. Relevant equation numbers are given on the right-hand side.

$\dot{D}_{ij}(\boldsymbol{r}) + D_{ijk}(\boldsymbol{r})_{\vert k} = -T_{ij}(\boldsymbol{r}) + 2\nu D_{ij}(\boldsymbol{r})_{\vert kk} - E_{ij}(\boldsymbol{r})$	(5), (6*a*–*d*)
$\dot{D}_{ii}(\boldsymbol{r}) + D_{iik}(\boldsymbol{r})_{\vert k} = 2\nu D_{ii}(\boldsymbol{r})_{\vert kk} - 4\epsilon$	(37)
$D_{ijk}(\boldsymbol{r})_{\vert jk} = -T_{ij}(\boldsymbol{r})_{\vert j}$	(39)
$E_{ii}(\boldsymbol{r}) = 4\epsilon$	(36)

TABLE 3. Relationships, in addition to those in table 1, proven for the anisotropic, locally homogeneous velocity field $\boldsymbol{u}$ that satisfies the Navier–Stokes equation and incompressibility. Relevant equation numbers are given on the right-hand side.

$D_{111}(r) = 6\nu\dfrac{\mathrm{d}D_{11}(r)}{\mathrm{d}r} - \dfrac{4}{5}\epsilon r$	(1)
$\dot{D}_{ij}(\boldsymbol{r}) + D_{ijk}(\boldsymbol{r})_{\vert k} = 2\nu D_{ij}(\boldsymbol{r})_{\vert kk} - \frac{4}{3}\epsilon\delta_{ij}$	(8)
See also text equations (41)–(44).	

TABLE 4. Relationships, in addition to those in tables 1, 2, and 3, proven for the locally homogeneous and locally isotropic velocity field $\boldsymbol{u}$ that satisfies the Navier–Stokes equation and incompressibility. For (1), local stationarity is also used. Relevant equation numbers are given on the right-hand side.

end of the Introduction, it is clear that local homogeneity is the approximation of retaining only the lowest-order term of such a Taylor series. The relaxation of turbulence toward local homogeneity can be studied by retaining the next non-vanishing terms in the Taylor series. These terms give rise to lowest-order derivatives with respect to $\boldsymbol{X}$. Again, the objective is to obtain expressions that can be evaluated by experiment.

Appendix

The curl operator with respect to $\boldsymbol{x}$ operating on index k is defined by $e_{ijk}\,\partial_j\, Y_{k\ldots}(\boldsymbol{x})$, where $Y_{k\ldots}(\boldsymbol{x})$ is a tensor of order 1 or greater, having k as one of its indices, and e_{ijk} is the permutation tensor, which is zero if any two indices are equal, and is $+1$ or -1 if the indices are respectively an even or odd permutation of 1, 2, and 3. The curl with respect to $\boldsymbol{r}$ operating on index j of the first term in (11) is

$$e_{nlj}\langle (V_i - V_i')\,(\partial_j S - \partial_j' S')\rangle_{|l} = \langle (e_{nlj}\,\partial_l\, V_i)\,(\partial_j S - \partial_j' S')\rangle \quad \text{(A 1)}$$

$$= e_{nlj}\langle V_i(\partial_j S - \partial_j' S')\rangle_{|l} \quad \text{(A 2)}$$

$$= -\langle (e_{nlj}\,\partial_l'\, V_i)\,(\partial_j S - \partial_j' S')\rangle \quad \text{(A 3)}$$

$$= 0.$$

Steps (A 1)–(A 3) use the fact that the curl of a gradient is identically zero and use local homogeneity to commute the derivatives into and out of the average. If the curl operates twice on (11), once on each of the indices, then both terms in (11) vanish. Therefore, on the basis of local homogeneity,

$$e_{nlj}\, e_{mki}\, T_{ij}(\boldsymbol{V}, S; \boldsymbol{r})_{|lk} = 0. \quad \text{(A 4)}$$

For brevity, the operation on the left-hand side of (A 4) is called the double curl. The left-hand side of (A 4) is symmetric under interchange of n and m and therefore constitutes six conditions, but setting $n = m$ and summing gives

$$T_{ii}(\boldsymbol{V}, S; \boldsymbol{r})_{|kk} - T_{ij}(\boldsymbol{V}, S; \boldsymbol{r})_{|ij} = 0,$$

which is already known on the basis of (17) and (19). The number of independent conditions imposed by (A 4) depends on the type of anisotropy (e.g. axisymmetry, local isotropy, etc.), with no independent conditions being added to (17) and (19) for the case of local isotropy.

The double curl of $D_{ij}(\boldsymbol{r})$ is

$$\Omega_{nm}(\boldsymbol{r}) \equiv -\langle \omega_n\, \omega_m' + \omega_n'\, \omega_m\rangle, \quad \text{(A 5)}$$

where $\boldsymbol{\omega}$ is the vorticity, and local homogeneity was used. The double curl of (5) is

$$\dot{\Omega}_{nm}(\boldsymbol{r}) + e_{nlj}\, e_{mpi}\, D_{ijk}(\boldsymbol{r})_{|klp} = 2\nu \Omega_{nm|kk}, \quad \text{(A 6)}$$

where (A 4) and (A 5) were used. For stationary turbulence, very large Reynolds numbers, and r very much greater than Kolmogorov's microscale, (A 6) gives

$$e_{nlj}\, e_{mpi}\, D_{ijk}(\boldsymbol{r})_{|klp} \approx 0. \quad \text{(A 7)}$$

Setting $n = m$ and summing over m in (A 7) gives $D_{iik}(\boldsymbol{r})_{|kll} \approx 0$; this is consistent with performing the Laplacian on (38). Analogous to the discussion following (A 4), the number of conditions imposed by (A 7) on the approach to local isotropy of $D_{ijk}(\boldsymbol{r})$ depends on the type of anisotropy that exists in a particular dataset.

REFERENCES

BATCHELOR, G. K. 1946 The theory of axisymmetric turbulence. *Proc. R. Soc. Lond.* A **186**, 480–502.

BATCHELOR, G. K. 1947 Kolmogoroff's theory of locally isotropic turbulence. *Proc. Camb. Phil. Soc.* **43**, 533–559.

BATCHELOR, G. K. 1951 Pressure fluctuations in isotropic turbulence. *Proc. Camb. Phil. Soc.* **47**, 359–374.

BORUE, V. & ORSZAG, S. A. 1996 Numerical study of three-dimensional Kolmogorov flow at high Reynolds numbers. *J. Fluid Mech.* **306**, 293–323.

FRISCH, U. 1995 *Turbulence, The Legacy of A. N. Kolmogorov*. Cambridge University Press.

HILL, R. J. 1993 Relationships between several fourth-order velocity statistics and the pressure structure function for isotropic, incompressible turbulence. *NOAA Tech. Rep.* ERL 449-ETL 65. (Order no. PB94-187309. Available from the author or the National Technical Information Service, 5285 Port Royal Rd., Springfield, VA 22161, USA.)

HILL, R. J. & WILCZAK, J. M. 1995 Pressure structure functions and spectra for locally isotropic turbulence. *J. Fluid Mech.* **296**, 247–269.

KÁRMÁN, T. VON & HOWARTH, L. 1938 On the statistical theory of isotropic turbulence. *Proc. R. Soc. Lond.* A **164**, 192–215.

KOLMOGOROV, A. N. 1941 Energy dissipation in locally isotropic turbulence. *Dokl. Akad. Nauk. SSSR* **32**, 16–18.

LINDBORG, E. 1996 A note on Kolmogorov's third-order structure function law, the local isotropy hypothesis and the pressure-velocity correlation. *J. Fluid Mech.* **326**, 343–356.

MONIN, A. S. 1959 The theory of locally isotropic turbulence. *Dokl. Akad. Nauk. SSSR* **125**, 515–518.

MONIN, A. S. & YAGLOM, A. M. 1975 *Statistical Fluid Mechanics: Mechanics of Turbulence*, Vol. 2. The MIT Press.

NELKIN, M. 1994 Universality and scaling in fully developed turbulence. *Adv. Phys.* **43**, 143–181.

OBUKHOV, A. M. 1949 Pressure fluctuations in turbulent flow. *Dokl. Akad. Nauk. SSSR* **66**, 17–20.

PUMIR, A. 1996 Turbulence in homogeneous shear flow. *Phys. Fluids* **8**, 3112–3127.

PUMIR, A. & SHRAIMAN, B. I. 1995 Persistent small-scale anisotropy in homogeneous shear flow. *Phys. Rev. Lett.* **75**, 3114–3117.

SADDOUGHI, S. G. & VEERAVALLI, S. V. 1994 Local isotropy in turbulent boundary layers at high Reynolds number. *J. Fluid Mech.* **268**, 333–372.